Lecture 4

Nevertheless, it is not trivial to solve the $\bar\partial$-equation and it is only solvable in certain situations, as we shall discuss. But there is a simple situation where we can solve $\bar\partial$ easily.

Thm 3. Let $f = \sum f_j \, d\bar z_j$ be a $\underbrace{(0,1)}_{\mathcal{C}^k \ (k \geq 1)}$ form in $\mathbb{C}^n$ w/ compact support. If $\bar\partial f = 0$ and $\underline{n > 1}$, then $\exists\, u \in \mathcal{C}^k$ w/ compact support s.t. $\bar\partial u = f$.

Let's first discuss the case $n = 1$. In this case, the condition $\bar\partial f = 0$ is vacuous. Nevertheless, you can solve $\bar\partial u = f$, but <u>not w/ compact support</u>.

To show how we can solve $\bar\partial u = f$ when $f = f_1 \, d\bar z$ and f_1 has compact support in, say, $K \subset\subset \mathbb{C}$, We need the Strong Cauchy Integral Formula. (SCIF)

Thm 4. Let $\Omega \subseteq \mathbb{C}$ w/ $\mathcal{C}^1$ smooth boundary (finitely many $\mathcal{C}^1$ Jordan curves) and $u \in \mathcal{C}^1(\overline{\Omega})$. Then, $\forall z \in \Omega$,

$$u(z) = \frac{1}{2\pi i}\left(\int_{\partial\Omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta + \int_{\Omega} \frac{\partial u}{\partial \bar\zeta} \frac{1}{\zeta - z}\, d\zeta \wedge d\bar\zeta \right)$$

Pf. Note that $d\left(\frac{u(\zeta)}{\zeta - z}\, d\zeta\right) = \frac{\partial u}{\partial \bar\zeta}\frac{1}{\zeta - z}\, d\bar\zeta \wedge d\zeta$ for $\zeta \in \overline{\Omega} \setminus \{z\}$. By Stokes Thm,

$B(z;\varepsilon) = \{\zeta : |\zeta - z| < \varepsilon\}$

$$\int_{\Omega \setminus B(z,\varepsilon)} \frac{\partial u}{\partial \bar\zeta}\frac{1}{\zeta - z}\, d\bar\zeta \wedge d\zeta = \int_{\partial\Omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta - \underbrace{\int_{\partial B(z,\varepsilon)} \frac{u(\zeta)}{\zeta - z}\, d\zeta}_{\to\, -2\pi i\, u(z) \text{ as } \varepsilon \to 0.}$$

$\blacksquare$

Now, we solve $\bar{\partial} u = f$ by defining

$$u(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta} = \left\{ \begin{array}{l} \zeta - z = w \\ d\zeta = dw \end{array} \right\}$$

$$= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_1(z+w)}{w} dw \wedge d\bar{w}.$$

$\Rightarrow$ Since $\frac{1}{w} \in L^1$ on a bdd open set Ω containing $K := \operatorname{supp} f_1$, we may integrate under $\int$ sign $\Rightarrow$

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2\pi i} \int_{\Omega} \frac{\partial f_1}{\partial \bar{z}} (z+w) \frac{1}{w} dw \wedge d\bar{w}$$

$$= \frac{1}{2\pi i} \int_{\Omega} \frac{\partial f_1}{\partial \bar{\zeta}}(\zeta) \frac{1}{\zeta - z} d\zeta \wedge d\bar{\zeta}$$

But by SCIF, since $f_1 \equiv 0$ outside $K \subset\subset \Omega \Rightarrow \int_{\partial\Omega} \frac{f_1(\zeta)}{\zeta - z} d\zeta = 0 \Rightarrow$

$$\frac{1}{2\pi i} \int_{\Omega} \frac{\partial f_1}{\partial \bar{\zeta}}(\zeta) \frac{1}{\zeta - z} d\zeta \wedge d\bar{\zeta} = f_1(z).$$

Thus, w/ $u(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f_1(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$

$\Rightarrow \frac{\partial u}{\partial \bar{z}} = f_1$ or $\bar{\partial} u = f$ as desired.

Note, no reason for u to have compact support, and one can show that this is not so in general.

Let's now prove Thm 3.

Pf. We try to imitate the pf for $n=1$. We now have an overdetermined system

$$\frac{\partial u}{\partial \bar{z}_k} = f_k \quad , \quad k = 1, \dots, n$$

where $f = \sum f_k d\bar{z}_k$ and $\bar{\partial} f = 0$. Set

$$u(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f_1(\zeta, z_2, \dots, z_n)}{\zeta - z_1}\, d\zeta \wedge d\bar{\zeta}.$$

$$= \frac{1}{2\pi i}\int f_1 \frac{(w + z_1, z_2, \dots, z_n)}{w}\, dw \wedge d\bar{w}$$

$\Rightarrow$

$$\frac{\partial u}{\partial \bar{z}_1} = \frac{1}{2\pi i}\int \frac{\partial f_1}{\partial \bar{z}_1}(w + z_1, \dots z_n)\, \frac{1}{w}\, dw \wedge d\bar{w}$$

$$= \frac{1}{2\pi i}\int \frac{\partial f_1}{\partial \bar{\zeta}}(\zeta, z_2, \dots, z_n)\, \frac{1}{\zeta - z_1}\, d\zeta \wedge d\bar{\zeta}$$

As in $n=1$, SCIF $\Rightarrow$

$$\frac{\partial u}{\partial \bar z_1} = f_1.$$

Next, recall that $\bar\partial f = 0 \Leftrightarrow \frac{\partial f_k}{\partial \bar z_j} = \frac{\partial f_j}{\partial \bar z_k}$

Thus, applying $\frac{\partial}{\partial \bar z_k}$ to u, we get (for $k \geq 2$)

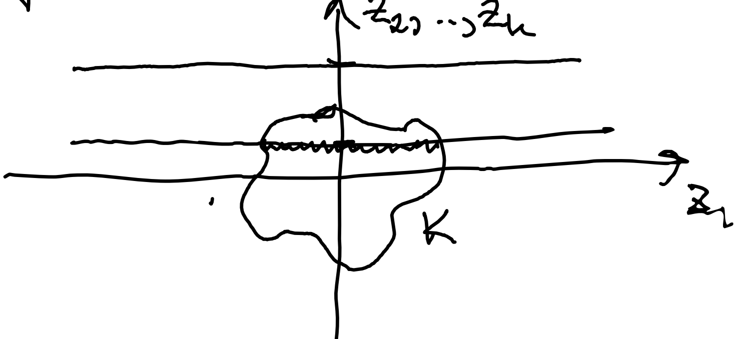
$$\frac{\partial u}{\partial \bar z_k} = \frac{1}{2\pi i} \int \frac{\partial f_1}{\partial \bar z_k}(\zeta, z_2, \dots, z_n) \frac{1}{\zeta - z_1} d\zeta \wedge d\bar\zeta$$

$$= \frac{1}{2\pi i} \int \frac{\partial f_k}{\partial \bar\zeta}(\zeta, z_2, \dots, z_n) \frac{1}{\zeta - z_1} d\zeta \wedge d\bar\zeta$$

$$= f_k$$

Therefore, u does satisfy $\bar\partial u = f$ as required.

Why does u have compact support.



For large $|z_2|^2 + \dots + |z_n|^2$, the domain of integration does not intersect the support of f.